LAWRENCE
LIVERMORE
NATIONAL
LABORATORY

UCRL-CONF-209972

The Comparison and Evaluation of Three Fiber Composite Failure Criteria

R. M. Christensen

February 25, 2005

SEM Annual Conference & Exposition on Experimental & Applied Mechanics
Portland, OR, United States
June 7, 2005 through June 9, 2005

Disclaimer

The Comparison and Evaluation of Three Fiber Composite Failure Criteria

Richard M. Christensen
Lawrence Livermore National Laboratory, P.O. Box 808 L-356, Livermore, CA and
Stanford University, Stanford, CA

ABSTRACT

Three specific failure criteria for the transversely isotropic fiber composite case will be discussed. All three use the polynomial expansion method. The three criteria are the Tsai-Wu criterion, the Hashin criterion and the Christensen criterion. All three criteria will be given in forms that admit direct and easy comparison, which has not usually been done. The central differences between these three criteria will be discussed, and steps will be taken toward the evaluation of them.

Background

That the field of composite material failure characterization is somewhat controversial should not be too surprising, considering the status of failure criteria development for the much simpler case isotropic materials. The two most common criteria for isotropic materials are the Mises and the Tresca forms, but these only apply to very ductile, isotropic metals. Both the Mises and Tresca criteria give yield levels in uniaxial tension and compression as being the same and both are independent of a superimposed mean normal stress, pressure. Neither of these ideal characteristics are found to occur with composites or even for many or most isotropic materials. In the case of non-ideal, isotropic materials there has been recent progress, Christensen [1], so too for anisotropic materials much has been done and there may be a reasonable expectation for further progress. The approach to be followed here is to employ a generalized method, applicable to any material class and then begin to specialize the symmetry class until we reach the form applicable to a fiber-matrix combination representative of behavior of an aligned fiber system at the lamina level or equivalent scale. The general approach is known as the polynomial formalism

Polynomial Expansion, No Symmetry

Polynomials provide the basis for many representational forms in mathematics. Interest here is in characterizing failure through the stress tensor, so tensor polynomials in σ_{ij} will be used. Actually this is the basis for obtaining strain energy representations and it will be used here for failure as well. All approaches to be shown here are of the polynomial expansion type. This general method provides an organized approach on the problem. Most other criteria appear to simply be directly postulated forms that are then compared with data or expectations.

Express the possible failure criterion as the polynomial

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j + F_{ijk}\sigma_i\sigma_j\sigma_k + ------ = 1 \tag{1}$$

where the contracted tensor notation is used with the coefficients F_i, F_{ij}, F_{ijk}....to be determined to give the best representation of relevant experimental data. The form of F_i is that of a second order tensor, as

$$F_i\sigma_i = \tilde{F}_{jk}\sigma_{jk}$$

Thus F_i has six independent components. Similarly F_{ij} is a fourth order tensor with 21 independent components. All higher-order tensors follow the same general character. The polynomial expansion (1) is normally truncated at the second-degree terms. Thus the failure form (1) at the second degree level contains twenty seven individual, independent parameters.

Under plane stress conditions with $\sigma_3 = \sigma_4 = \sigma_5 = 0$ then (1) is given by

$$\begin{aligned}&\left[F_1\sigma_1 + F_2\sigma_2 + F_6\sigma_6\right] + \\ &\left[F_{11}\sigma_1{}^2 + F_{22}\sigma_2{}^2 + F_{66}\sigma_6{}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{16}\sigma_1\sigma_6 + 2F_{26}\sigma_2\sigma_6\right] + ------ = 1\end{aligned} \tag{2}$$

The plane stress form (2) involves nine independent parameters at the second-degree level.

Most interests are at the three dimensional level, and twenty-seven experiments to determine properties is an almost unthinkable task. Reductions in the numbers of parameters are afforded by symmetry restrictions in particular cases, and by specific physical restrictions.

<u>Orthotropy</u>. Consider the case of orthotropy and take the coordinate planes parallel to the symmetry planes. The term $F_i\sigma_i$ is written out as

$$F_i\sigma_i = F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3 + F_4\sigma_4 + F_5\sigma_5 + F_6\sigma_6$$

The shear stress terms involving σ_4, σ_5, and σ_6 give a physically unacceptable effect due to the sign of the shear stress, unless the related coefficients vanish, as

$$F_4 = F_5 = F_6 = 0$$

This simplification would not be allowed if the material symmetry planes were not parallel to the coordinate planes. Now, in the second term of (1), for the coupling terms between normal stress and shear stress to be independent of the sign of the shear stress, it again follows that

$$F_{14} = F_{15} = F_{16} = 0 \ , \ F_{24} = F_{25} = F_{26} = 0 \ , \ F_{34} = F_{35} = F_{36} = 0$$

Furthermore, the shear strengths are assumed to be uncoupled; thus to be independent of the sign of the shear stress

$$F_{45} = F_{46} = F_{56} = 0$$

Using these forms in (1) it becomes

$$\begin{aligned}&\left(F_1\sigma_1 + F_2\sigma_2 + F_3\sigma_3\right) + \left(F_{11}\sigma_1{}^2 + F_{22}\sigma_2{}^2 + F_{33}\sigma_3{}^2 + 2F_{12}\sigma_1\sigma_2 + 2F_{23}\sigma_2\sigma_3\right. \\ &\left.+2F_{13}\sigma_1\sigma_3 + F_{44}\sigma_4{}^2 + F_{55}\sigma_5{}^2 + F_{66}\sigma_6{}^2\right) + ------ = 1\end{aligned} \tag{3}$$

Thus orthotropy contains twelve parameters or constants at the second-degree level.

For plane stress we have by definition

$$\sigma_3 = \sigma_4 = \sigma_5 = 0$$

leaving (3) as

$$\left(F_1\sigma_1 + F_2\sigma_2\right) + \left(F_{11}\sigma_1{}^2 + F_{22}\sigma_2{}^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6{}^2\right) + ------ = 1 \tag{4}$$

Thus in this case F_i has two components and F_{ij} has four. At the level of truncation shown in (4) there are six constants to be determined.

Transverse Isotropy

The case of transverse isotropy is that usually taken for an aligned fiber composite material at the macroscopic lamina level. Transverse isotropy is a special case of orthotropy. Take the x_2x_3 plane to be the plane of symmetry. Then from symmetry for (1)

$$F_2 = F_3 \ , \ F_{12} = F_{13} \ , F_{22} = F_{33} \ , \ F_{55} = F_{66}$$

Also, the shear condition gives

$$F_{44} = 2\left(F_{22} - F_{23}\right)$$

These further restrictions on the orthotropic results reduce (3) to

$$\begin{aligned}&\left[F_1\sigma_1 + F_2\left(\sigma_2 + \sigma_3\right)\right] + \left[F_{11}\sigma_1{}^2 + F_{22}\left(\sigma_2{}^2 + \sigma_3{}^2\right) + 2F_{12}\left(\sigma_1\sigma_2 + \sigma_1\sigma_3\right)\right. \\ &\left.+2F_{23}\sigma_2\sigma_3 + 2\left(F_{22} - F_{23}\right)\sigma_4{}^2 + F_{55}\left(\sigma_5{}^2 + \sigma_6{}^2\right)\right] + ------ = 1\end{aligned} \tag{5}$$

There are now a total of seven material parameters to be determined at the level shown. For a plane stress condition the number of independent parameters is reduced to six, the same as orthotropy under plane stress.

In the following, three specific failure criteria for the transversely isotropic fiber composite case will be given. All three use the polynomial expansion method already given. The three criteria and the associated numbers of parameters are:

Tsai-Wu	7 Parameters
Hashin	6 Parameters
Christensen	5 Parameters

These criteria will be stated in the chronological order of their development, spreading out over about thirty years.

There are many theories with many more parameters than will be considered in the three forms to be given. Such theories will not be covered since they usually become curve fitting exercises. There are theories with fewer parameters than covered here but they don't seem to have enough range and texture to cover the many complex effects than can occur. The purpose in giving these three main stream theories rather than just one is to show the variety of effects and interpretations that are possible.

The experimental information that may be available to evaluate the parameters in the failure criteria are the following:

T_{11} and C_{11}	Fiber direction uniaxial tensile and compressive strengths
T_{22} and C_{22}	Transverse uniaxial tensile and compressive strengths
S_{12} and S_{23}	Fiber direction and transverse shear strengths

All of these comprise one dimensional stress state experiments. Any further testing information that may be needed must come from multi-axial experiments, which have proven to be difficult to obtain, at least on a routine basis. It might be added that the transverse shear strength, S_{23} involves a difficult experiment that is not usually reported.

In the following criteria, the polynomial expansion is always truncated at the second-degree terms. Using third degree terms has been examined, Tennyson et. al. [2], but has not proven to be particularly useful.

<u>Tsai-Wu Criterion</u>

The Tsai-Wu [3] criterion follows directly from the polynomial expansion (5), with each parameter requiring a separate experimental evaluation. In direct notation and after some consolidation of terms, the Tsai-Wu form for the safety domain becomes

$$
\left(\frac{1}{T_{11}}-\frac{1}{C_{11}}\right)\sigma_{11}+\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)\left(\sigma_{22}+\sigma_{33}\right)+\frac{\sigma_{11}{}^{2}}{T_{11}C_{11}}
$$
$$
+\frac{1}{T_{22}C_{22}}\left(\sigma_{22}+\sigma_{33}\right)^{2}+2F_{12}\sigma_{11}\left(\sigma_{22}+\sigma_{33}\right) \qquad (6)
$$
$$
+\frac{1}{S_{23}{}^{2}}\left(\sigma_{23}{}^{2}-\sigma_{22}\sigma_{33}\right)+\frac{1}{S_{12}{}^{2}}\left(\sigma_{12}{}^{2}+\sigma_{31}{}^{2}\right)\leq 1
$$

The seven parameters are given by the six experimental results shown plus F_{12}, which is called an interaction parameter. The latter must be evaluated from multi-axial data or often it is estimated. In the original form of the Tsai-Wu criterion and in (5) there is a second interaction parameter, F_{23}, but it has necessarily been eliminated here by the transverse shear strength, S_{23}, and the other strength properties. In the third criterion, to be considered later, a no failure condition under compressive hydrostatic stress is imposed. If that same condition is imposed here upon (6), it determines F_{12} to be

$$
F_{12}=\frac{1}{4S_{23}{}^{2}}-\frac{1}{T_{22}C_{22}}-\frac{1}{4T_{11}C_{11}} \qquad (6a)
$$

All terms in (6) are fully interactive with each other, meaning all stress components are coupled together in an interactive manner. This is the most direct interpretation of the polynomial form (5). The following two methods give somewhat different interpretations of (5). The Tsai-Wu criterion is sometimes called the tensor polynomial criterion, but that terminology would be best used only in referring to the generic type because all three criteria given here are variations of the tensor polynomial type. In the case of isotropy the Tsai-Wu criterion (6) and condition (6a) admit reduction to the Mises criterion.

<u>Hashin Criterion</u>

The Hashin [4] criterion also begins with the second-degree polynomial expansion in (5). The failure modes are then decomposed into matrix controlled and fiber controlled groups, depending upon which stress components act upon the failure planes, these planes being taken parallel and perpendicular to the fiber direction, respectively. Also, the interaction parameter F_{12} in (5) is taken to vanish. Next, each mode is further decomposed into tensile controlled and compressive controlled forms, with several of the same terms appearing in each. This introduces four additional parameters, bringing the total parameter count to ten. Finally, four separate assumptions or conditions are imposed, bringing the total parameter count back to six. The tensile and compressive type matrix modes of failure are differentiated by the sign of the transverse direction mean normal stress.

The Hashin failure criterion is then given by:

Tensile Matrix Mode, $(\sigma_{22} + \sigma_{33}) > 0$

$$\frac{1}{T_{22}^{\;2}}(\sigma_{22} + \sigma_{33})^2 + \frac{1}{S_{23}^{\;2}}\left(\sigma_{23}^{\;2} - \sigma_{22}\sigma_{33}\right) + \frac{1}{S_{12}^{\;2}}\left(\sigma_{12}^{\;2} + \sigma_{31}^{\;2}\right) \le 1 \tag{7a}$$

Compressive Matrix Mode, $(\sigma_{22} + \sigma_{33}) < 0$

$$\begin{aligned} &\frac{1}{C_{22}}\left[\left(\frac{C_{22}}{2S_{23}}\right)^2 - 1\right](\sigma_{22} + \sigma_{33}) + \frac{1}{4S_{23}^{\;2}}(\sigma_{22} + \sigma_{33})^2 \\ &+ \frac{1}{S_{23}^{\;2}}\left(\sigma_{23}^{\;2} - \sigma_{22}\sigma_{33}\right) + \frac{1}{S_{12}^{\;2}}\left(\sigma_{12}^{\;2} + \sigma_{31}^{\;2}\right) \le 1 \end{aligned} \tag{7b}$$

Tensile Fiber Mode, $\sigma_{11} > 0$

$$\left(\frac{\sigma_{11}}{T_{11}}\right)^2 + \frac{1}{S_{12}^{\;2}}\left(\sigma_{12}^{\;2} + \sigma_{31}^{\;2}\right) \le 1 \tag{7c}$$

Compressive Fiber Mode, $\sigma_{11} < 0$

$$\left(\frac{\sigma_{11}}{C_{11}}\right)^2 \le 1 \tag{7d}$$

The two failure criteria, (6) and (7) although starting with the same polynomial terms in (5) certainly take very different final forms.

Failure Considerations

The two criteria just given were developed about twenty-five and thirty-five years ago and both are widely used. The sentinal differences between these two forms suggests some basic questions of relevance to failure criteria. The most obvious questions are the following.

i) Is it necessary or advisable to decompose failure into fiber controlled versus matrix controlled modes?

ii) If the failure is decomposed into fiber and matrix modes, is it then necessary or advisable to further decompose these into tensile versus compressive modes?

and

iii) What is the minimum number of parameters that would reasonably be expected in order to comprehensively quantify failure for fiber composites?

To pursue these questions, begin with the collection of terms that occur in the polynomial expansion of second degree for transverse isotropy. From (5) these seven terms can be written in terms of the invariants as

$$\sigma_{11},\ \sigma_{ii},\ \sigma_{11}{}^2,\ \sigma_{ii}{}^2,\ \sigma_{11}\ \sigma_{ii},\ \sigma_{1i}\ \sigma_{1i},\ \sigma_{ij}\ \sigma_{ij},\quad \text{i,j=2,3} \tag{8}$$

Next, these terms will be normalized by moduli, either E_{11}, E_{22} or μ_{12} as appropriate to the related stress components. This gives the same seven terms but now in preferred non-dimensional forms as

$$\frac{\sigma_{11}}{E_{11}},\ \frac{\sigma_{ii}}{E_{22}},\ \frac{\sigma_{11}{}^2}{E_{11}{}^2},\ \frac{\sigma_{ii}{}^2}{E_{22}{}^2},\ \frac{\sigma_{11}}{E_{11}}\ \frac{\sigma_{ii}}{E_{22}},\ \frac{\sigma_{1i}\sigma_{1i}}{\mu_{12}{}^2},\ \frac{\sigma_{ij}\sigma_{ij}}{E_{22}{}^2},\quad \text{i,j=2,3} \tag{9}$$

As a limiting case, now consider a fiber composite with infinitely stiff fibers. This mathematical abstraction is of special interest because it naturally brings out the matrix controlled failure modes that remain active. The corresponding terms in (9) with $E_{11} \to \infty$ are then

$$0,\ \frac{\sigma_{ii}}{E_{22}},\ 0,\ \frac{\sigma_{ii}{}^2}{E_{22}{}^2},\ 0,\ \frac{\sigma_{1i}\sigma_{1i}}{\mu_{12}{}^2},\ \frac{\sigma_{ij}\sigma_{ij}}{E_{22}{}^2},\quad \text{i,j=1,2} \tag{10}$$

The combination of the remaining terms in (10) gives the matrix controlled failure mode as

$$a\frac{\sigma_{ii}}{E_{22}} + b\frac{\sigma_{ii}{}^2}{E_{22}{}^2} + c\frac{\sigma_{1i}\sigma_{1i}}{\mu_{12}{}^2} + d\frac{\sigma_{ij}\sigma_{ij}}{E_{22}{}^2} = 1,\quad \text{i,j=2,3} \tag{11}$$

The moduli can be absorbed into the coefficients to give

$$a\sigma_{ii} + b\sigma_{ii}{}^2 + c\sigma_{1i}\sigma_{1i}, + d\sigma_{ij}\sigma_{ij} = 1 \quad \text{i,j=2,3} \tag{12}$$

where a, b, c and d are parameters to be determined and different from the like symbols in (11). The form (12) applies not only to the infinitely stiff fiber limiting case, but also to the contiguous range of very stiff fiber cases.

Now the complementary fiber controlled mode of failure will be found. Start with the terms in (9), but the matrix controlled terms in (9) must be eliminated, otherwise this second criterion would just repeat the one already found when $\sigma_{11} = 0$. Eliminating the terms in (11) from (9) then leaves the fiber controlled criterion as

$$e\sigma_{11} + f\sigma_{11}{}^2 + g\sigma_{11}\sigma_{ii} = 1 \quad \text{i=2,3} \tag{13}$$

where the moduli have been absorbed into the parameters e, f and g.

This separation into fiber versus matrix modes of failure is necessary and unavoidable in the infinitely stiff fiber case and it is physically consistent and compatible in the adjoining very stiff fiber range. The question then comes down to that of what constitutes a very stiff fiber system? There is no specific rule, but reasonable guidance would be that if the degree of anisotropy is an order of magnitude or greater, then the decomposition into fiber and matrix modes of failure is necessary, otherwise it is not necessary and not appropriate. This gives clear guidance on question i) above. Both forms, decomposed or not decomposed, have separate and distinct ranges of validity. The result that the two failure modes must decouple for highly anisotropic aligned fiber systems has a satisfying analogy with behavior at the next larger scale of effects. Specifically, for fiber composite laminates, the in-plane failure modes decouple from the delamination failure modes, where again highly anisotropic effects dominate.

Regarding questions ii) and iii) we turn to the case of isotropic materials. In recent work Christensen [1] showed that isotropic material failure can be characterized by just two parameters for a wide range of materials types. Two failure parameters is of the same number as the number of elastic properties for isotropic materials. This suggests that five or more parameters probably are needed to characterize failure for transversely isotropic materials, compared with its five elastic properties. Furthermore, in the above noted work for isotropic materials, it was found that there was no necessity or even advantage to decompose into tensile versus compressive modes of failure. The nature of a polynomial expansion up to second degree automatically brings in quadratic forms with two roots that naturally fall into tensile and compressive behaviors. It would be difficult to see how to do the isotropic case other than by this means.

Christensen Criterion

The third failure criterion, Christensen [5, 6], to be included here had the advantage (and benefit) of coming after the first two. In this sense, it complies with the failure considerations just discussed. The procedure starts with the polynomial expansion of second degree and for very stiff, highly anisotropic fiber systems the failure form is decomposed into the matrix and fiber controlled modes of failure (12) and (13). These forms then contain seven parameters. Next each of these is required to allow unlimited hydrostatic pressure without failure, which then reduces the parameter count by two. The resulting failure criterion is

Matrix Mode

$$\left(\frac{1}{T_{22}}-\frac{1}{C_{22}}\right)(\sigma_{22}+\sigma_{33})+\frac{1}{T_{22}C_{22}}\left[(\sigma_{22}-\sigma_{33})^2+4\sigma_{23}{}^2\right]+\frac{(\sigma_{12}{}^2+\sigma_{31}{}^2)}{S_{12}{}^2}\leq 1 \qquad (14a)$$

and

Fiber Mode

$$\left(\frac{1}{T_{11}}-\frac{1}{C_{11}}\right)\sigma_{11}+\frac{\sigma_{11}{}^{2}}{T_{11}C_{11}}-\frac{1}{4}\left(\frac{1}{T_{11}}+\frac{1}{C_{11}}\right)^{2}\sigma_{11}\left(\sigma_{22}+\sigma_{33}\right)\leq 1 \tag{14b}$$

There are five parameters or properties in (14). The transverse shear stress failure property, S_{23}, is not involved here, it was in effect eliminated by requiring the independence to hydrostatic pressure. If σ_{22} and σ_{33} are small compared with σ_{11} then fiber mode (14b) just becomes the maximum stress criterion in the fiber direction. In contrast, the Tsai-Wu criterion shows a much stronger interaction between fiber direction strength and transverse pressure than does (14b), while the Hashin criterion shows no interaction at all between them.

Overview of Failure

The three failure criteria just given show the variety of physical effects which can or may occur. These three approaches are interrelated, all being variations on the theme of a polynomial expansion. In terms of the number of parameters to be determined and the number of terms that interact in the failure criteria, the third criterion is the simplest of the three. In terms of approach and methodology, the third criterion is intermediate between the other two. All three criteria are serious, well considered efforts, and their differences reflect the complexity of the program to determine failure criteria.

The similarities of the three criteria are that they all show an asymmetry in uniaxially tensile and compressive strengths, and they all show a sensitivity to mean normal stress. Another example of their differences in addition to the one already mentioned is that of the manner in which fiber direction uniaxial stress σ_{11} interacts with fiber direction shear stress, σ_{12}. The Tsai-Wu criterion shows a strong interaction between these two stress components. The Hashin criterion has them interacting when σ_{11} is tensile but not when it is compressive. The Christensen criterion states that the shear stress σ_{12} has a negligible effect on the fiber direction strength, for very stiff fiber systems in which there is no rotation of the fiber direction.

There are many other theories of failure, for example a prominent one is that of Puck and Schürmann [7] which is based upon the Coulomb-Mohr approach. Probably there never will be a single, universally agreed upon theory of failure for fiber composites. There simply are too many very different points of view, amply sustained by the inherent complexity of the materials systems. Despite this diversity of opinion there still can be reasonably high standards and measures of quality in the effort to characterize failure for composites, but it does come down to a matter of individual preference and judgment.

This work was performed under the auspices of the U.S. Department of Energy by University of California, Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.

References

1. Christensen, R. M. "A two-property yield, failure (fracture) criterion for homogeneous, isotropic materials," J. Engr. Mats. And Technology, vol. 126, 45 (2004).
2. Tennyson, R. C., MacDonald, D., and Nanyaro, A. P., "Evaluation of the tensor polynomial failure criterion for composite materials," *J. Comp. Mater.*, vol. 12, 63 (1978).
3. Tsai, S. W. and Wu, E. M., "A general theory of strength for anisotropic materials," *J. Comp. Mater.*, vol. 5, 58 (1971).
4. Hashin, Z., "Failure criteria for unidirectional fiber composites," J. Appl. Mech., vol. 47, 329 (1980).
5. Christensen, R. M., "Stress based yield/failure criteria for fiber composites," Int. J. Solids Structures, vol. 34, 529 (1997).
6. Christensen, R. M., "The numbers of elastic properties and failure parameters for fiber composites," J. Engr. Mats. And Technology, vol. 120, 110 (1998).
7. Puck, A. and Schürmann , H., "Failure analysis of FRP laminates by means of physically based phenomenological models," Comps. Sci. and Tech., vol. 62, 1633. (2002).